EE-446 Antennas

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Practical information

- Useful information and documents on the Moodle site of the course

- Our emails: romain.fleury@epfl.ch, amir.jafargholi@epfl.ch, anja.skrivervik@epfl.ch

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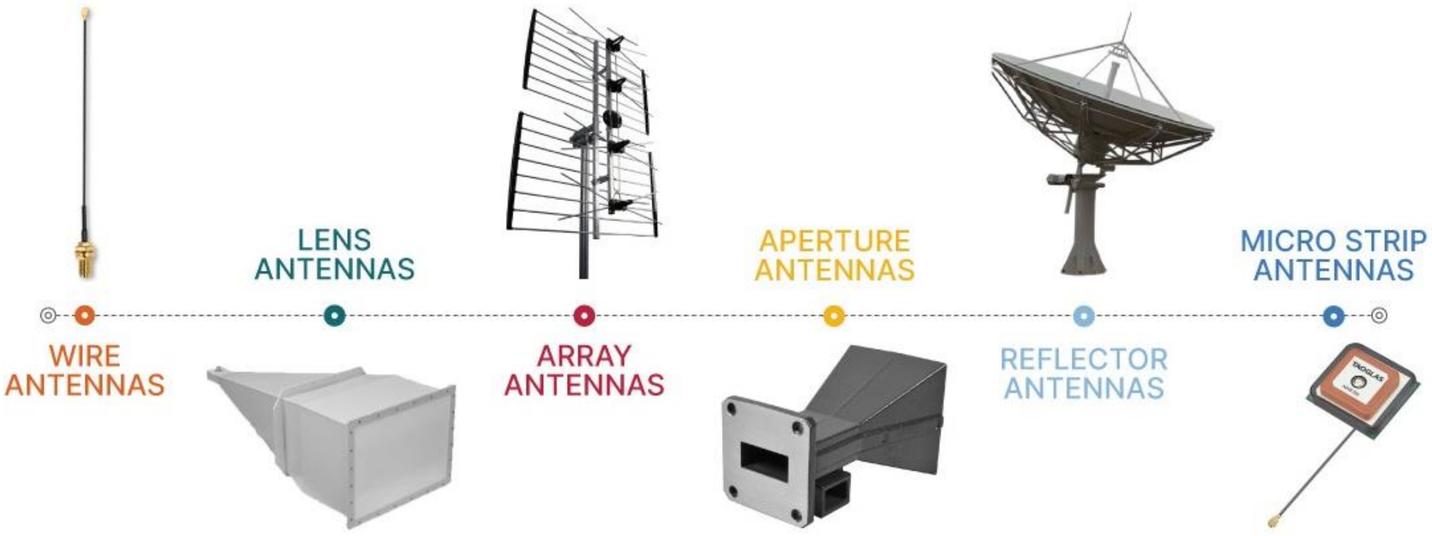
- Evaluation:
  - Midterm on the theoretical part: November 12th, 13:15- 16:00
  - Project presentations: December 17th, 13:15-17:15

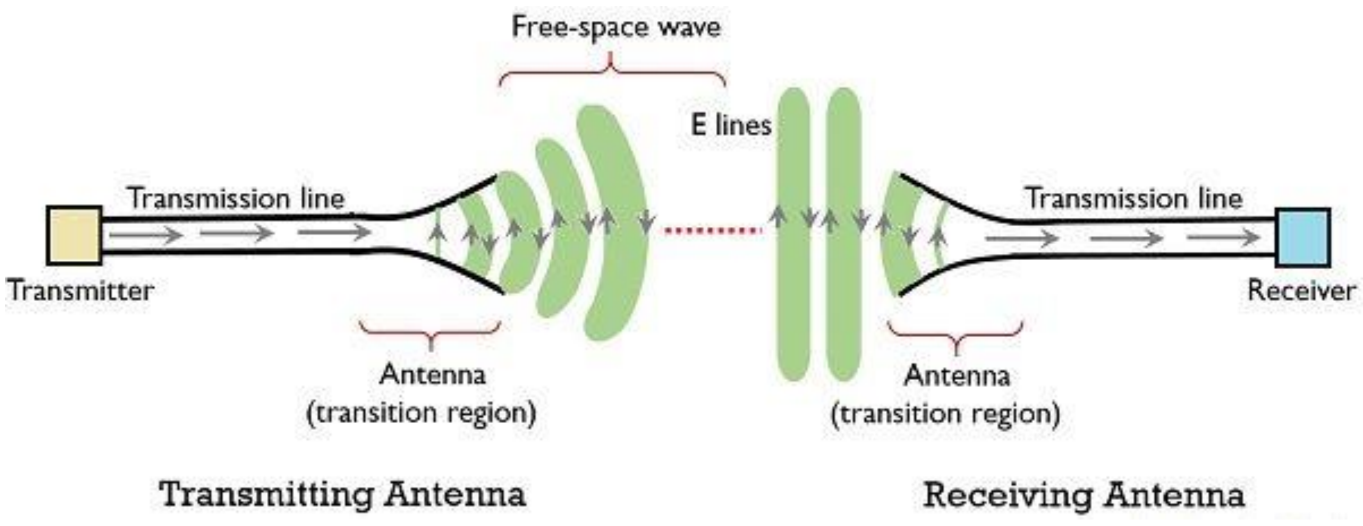
- Electromagnetic simulations will be used for exercises and for the project. It is important to install the simulator on your computer asap. The tutorial on Moodle helps you doing so. If you have difficulties installing the software, please contact the IT manager of the Microwave and Antenna Group, Jiajun Li jiajun.li@epfl.ch

What this course is about



An antenna (American English) or aerial (British English) is **an electronic device that converts an alternating electric current into radio waves**. *source Wikipedia (en), article "Antenna (radio)*

Antennas as transducers



Circuit theory
Voltage and current

Transmission-line theory
Voltage and current waves

Antenna
From circuit to free-space

Electromagnetic theory (Maxwell)
Electric and Magnetic fields

Example of Antenna science:
emitting antennas from the circuit side/excitation problem

Circuit theory
Voltage and current

Transmission-line theory
Voltage and current waves

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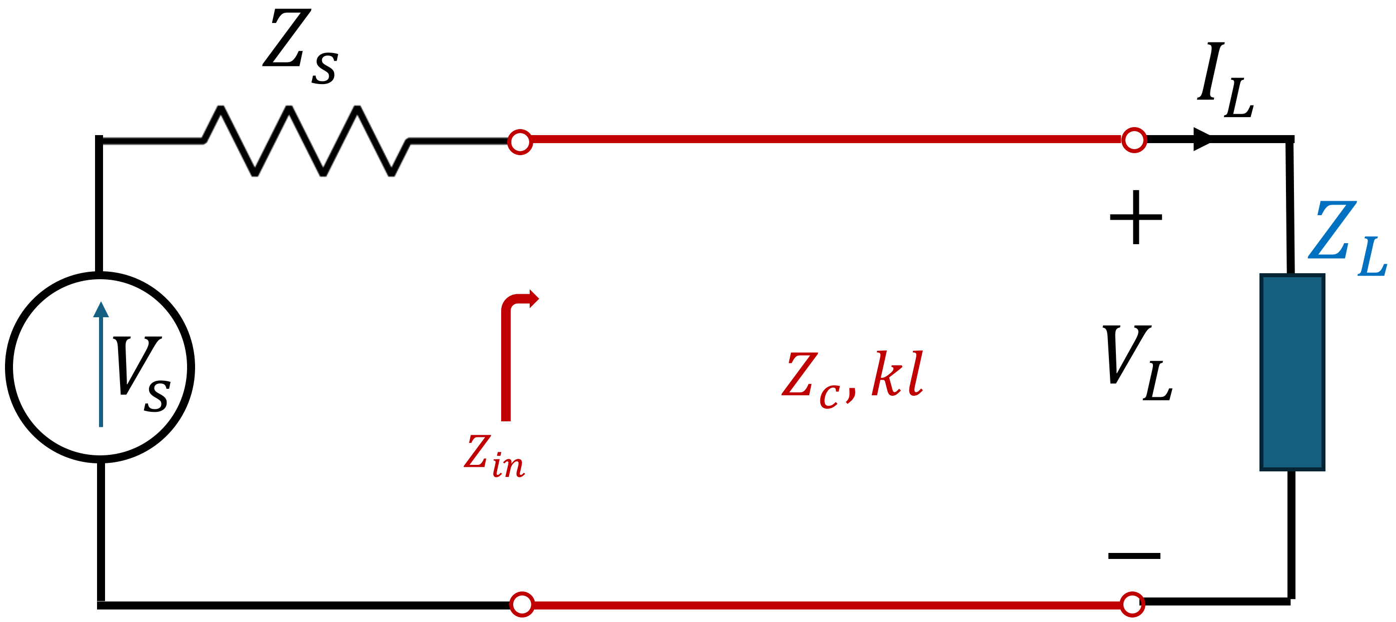
describe all this with an impedance $Z_L(\omega)$
requires measurement or solving Maxwell's equations



then use transmission line theory (cf. EE-200)
to calculate V_L and I_L:

$I_L = V_L/Z_L$

$V_L = V^+(1+\Gamma_L)$

$\Gamma_L = (Z_L - Z_c)/(Z_L + Z_c)$

$$V^+ = \frac{Z_{in}V_s}{(Z_{in}+Z_s)e^{jkl}(1+\Gamma_L e^{-2jkl})}$$

$$Z_{in} = Z_c \frac{Z_L + jZ_c \tan kl}{Z_c + jZ_L \tan kl}$$

impedance matching: $Z_L = Z_c$
reduce reflection

conjugate matching: $Z_{in} = Z_S^*$
more power out of source

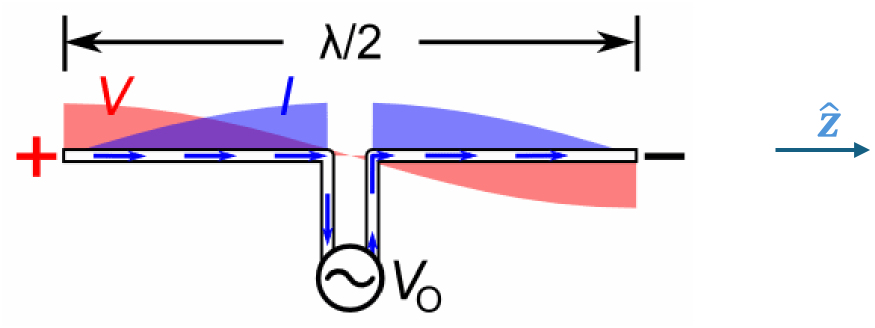
Example of antenna science: Antennas from the free-space side/the radiation problem

Circuit theory
Voltage and current

Transmission-line theory
Voltage and current waves

Assume you know the strength of the radiating current I_L
and its distribution (guessed or simulated)

$J(\boldsymbol{r}') = \hat{\boldsymbol{z}}\, I_L \cos\left(\frac{2\pi x}{\lambda}\right)\delta(y)\delta(z)$ when $|x| < \lambda/4$



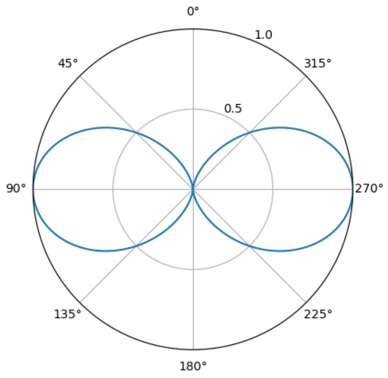
half wave antenna

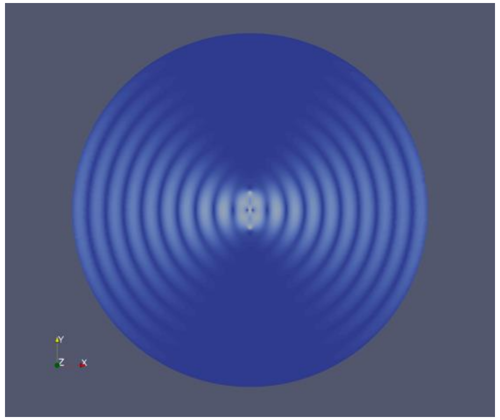
Antenna
From circuit to free-space

Electromagnetic theory (Maxwell)
Electric and Magnetic fields

then use far-field integrals to calculate the electromagnetic field in all directions (we'll learn this!)

$$E_\theta = Z_0\, I_L \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\sin\left(\frac{\pi}{2} - kr\right)$$





Course outline

Part I : Learn the basics (weeks 1-7)

1. Reminder of Electromagnetic theory (2 hours): Maxwell equations and important results
2. Antenna definitions: gain, directivity, Friis formula, efficiency (2 hours)
3. System aspect of Antennas: link budget, SNR, gain (2 hours)
4. Antenna families: patch, apertures, horn, slot and arrays (8 hours)

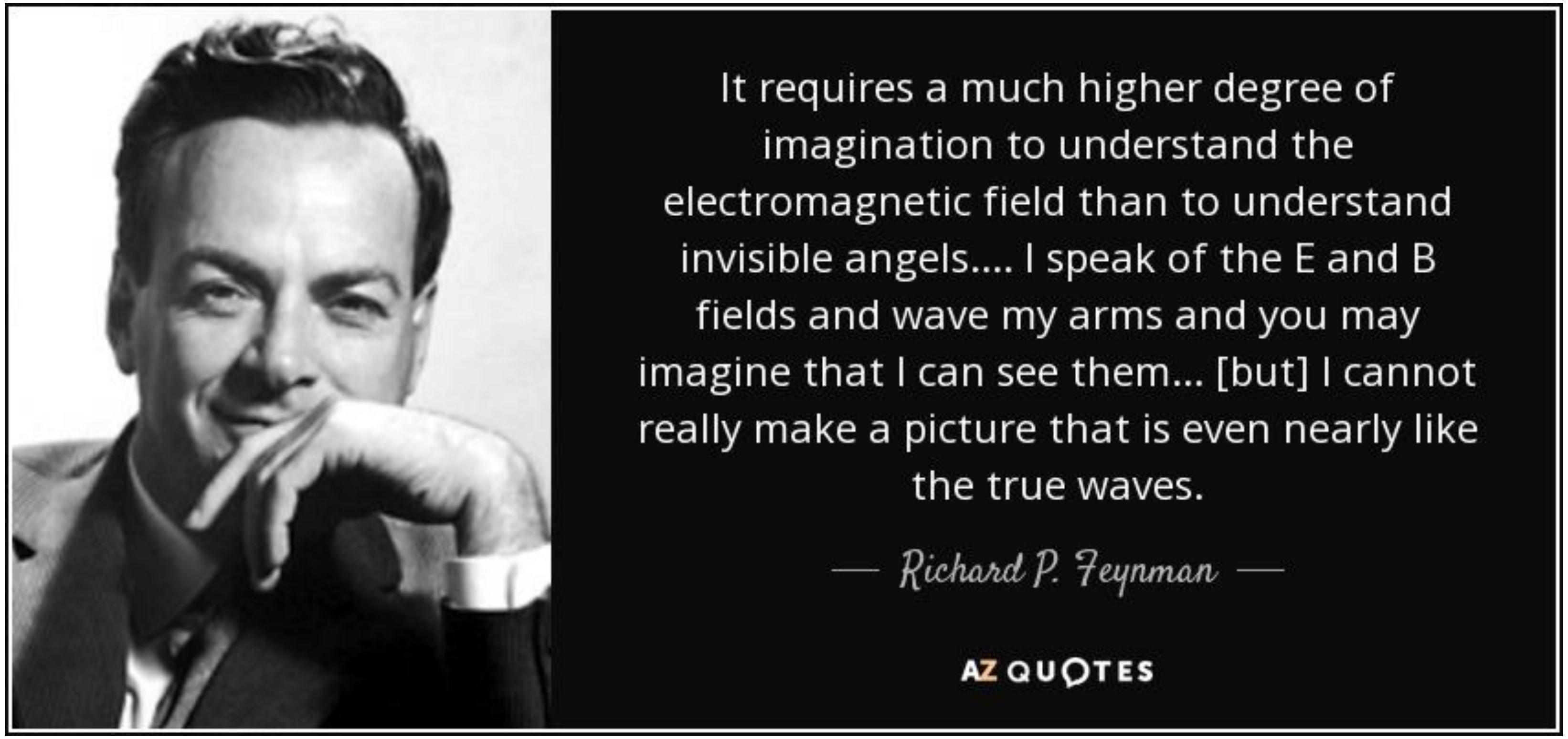
Exam

Part II : Apply your knowledge (weeks 8-14)

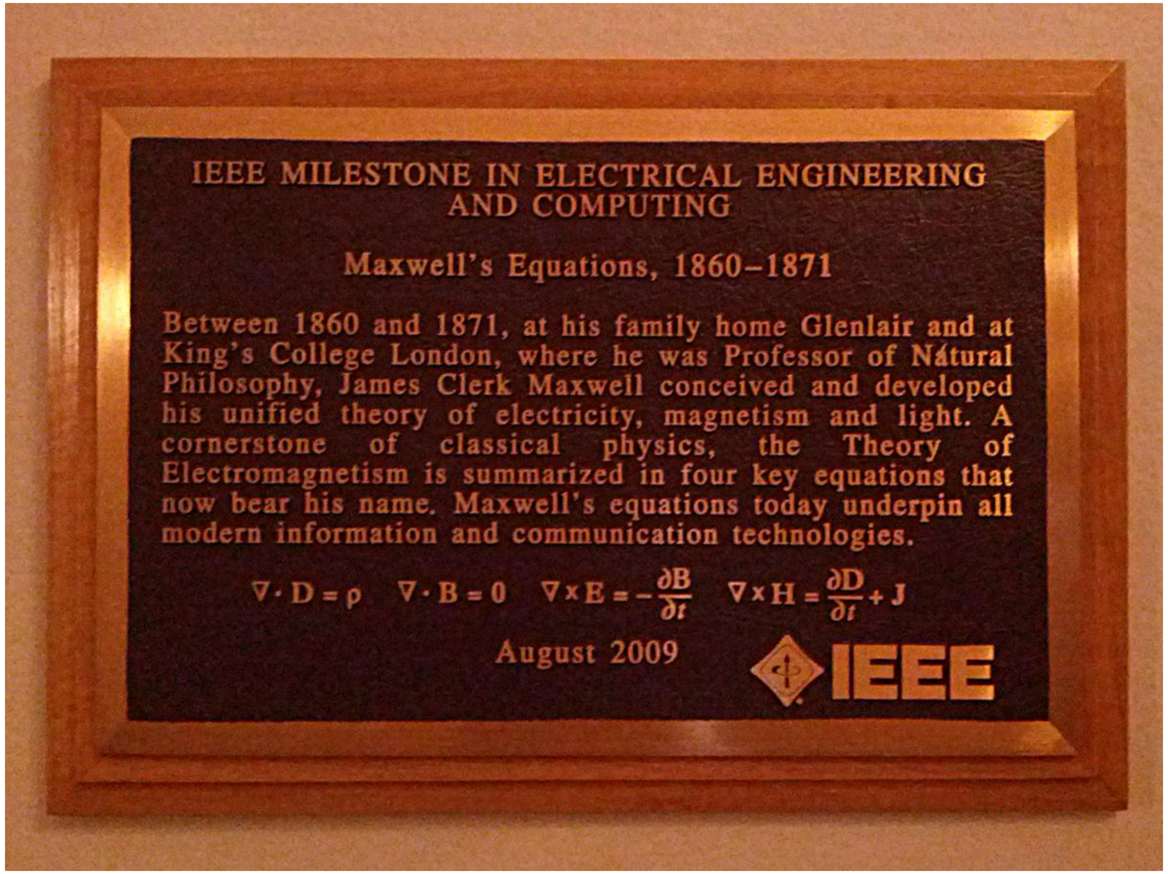
1. Work in groups
2. Design an antenna
3. Build it
4. Test it

Presentations and brief report

Reminders on Electromagnetic theory



1) Reminders on Maxwell’s equations



Postulates of Electromagnetic theory

Classical mechanics

$$m\boldsymbol{a} = \sum \boldsymbol{F_{ext}}$$

Lorentz force

$\boldsymbol{F} \quad = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ force on a particle with charge q [C] and velocity $\boldsymbol{v}$ [m/s]

$\boldsymbol{E}$: electric field [V/m]

$\boldsymbol{B}$: magnetic flux density [T or Wb/m]

Maxwell's equations (in vacuum)

$\nabla . \boldsymbol{D} = \rho$

$\nabla . \boldsymbol{B} = 0$

$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$

$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$

$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$: Electric flux density [C/m]

$\rho = \sum_i \ \rho_i \delta(r - r_i)$: electric charge density [C/m^3]

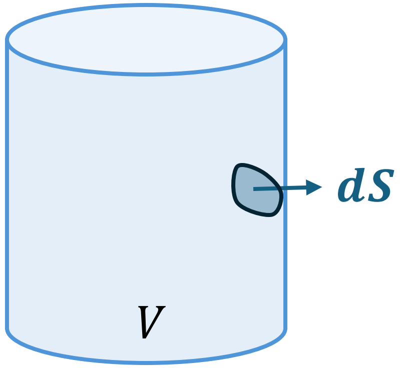
$\frac{\boldsymbol{B}}{\mu_0} = \boldsymbol{H}$: Magnetic field [A/m]

$\boldsymbol{J} = \sum_i \ \rho_i \boldsymbol{v_i} \delta(r - r_i)$: electric current density [A/m^2]

Two useful mathematical theorems

divergence theorem:

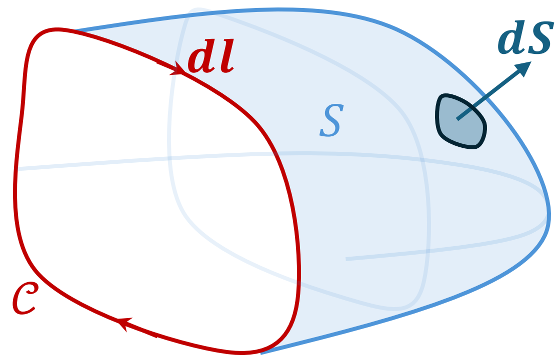
$$\iiint_V \nabla.\boldsymbol{u}\, dV = \iint_S \boldsymbol{u}.\boldsymbol{dS}$$



Stokes theorem:

$$\int_{\mathcal{C}} \boldsymbol{u}.\boldsymbol{dl}$$

$$= \iint_S (\nabla \times \boldsymbol{u}).\boldsymbol{dS}$$



Integral Maxwell equations

Differential (local) form:

$$\nabla . \boldsymbol{D} = \rho$$

$$\nabla . \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Integral form, on closed S or $\mathcal{C}$:

$$\iint_S \boldsymbol{D} . \boldsymbol{dS} = \boldsymbol{Q_{int}}$$

$$\iint_S \boldsymbol{B} . \boldsymbol{dS} = \boldsymbol{0}$$

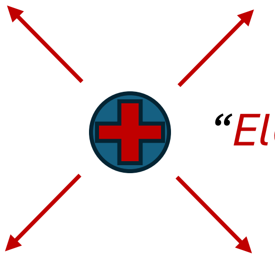
$$\int_{\mathcal{C}} \boldsymbol{E} . \boldsymbol{dl} = -\frac{\partial}{\partial t} \iint_S \boldsymbol{B} . \boldsymbol{dS}$$

$$\int_{\mathcal{C}} \boldsymbol{H} . \boldsymbol{dl} = \boldsymbol{I} + \frac{\partial}{\partial t} \iint_S \boldsymbol{D} . \boldsymbol{dS}$$

Interpretation

$\nabla . \boldsymbol{D} = \rho$

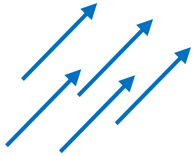
Maxwell-Gauss



"Electric field diverges only around charges"

$\nabla . \boldsymbol{B} = 0$

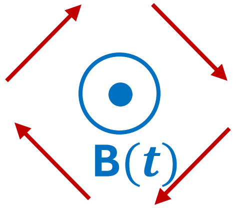
Magnetic flux law



"Magnetic field never diverges"

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

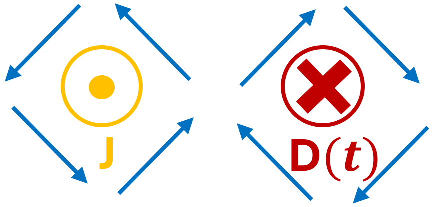
Maxwell-Faraday



"Electric field circulates around a time-varying magnetic field"

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

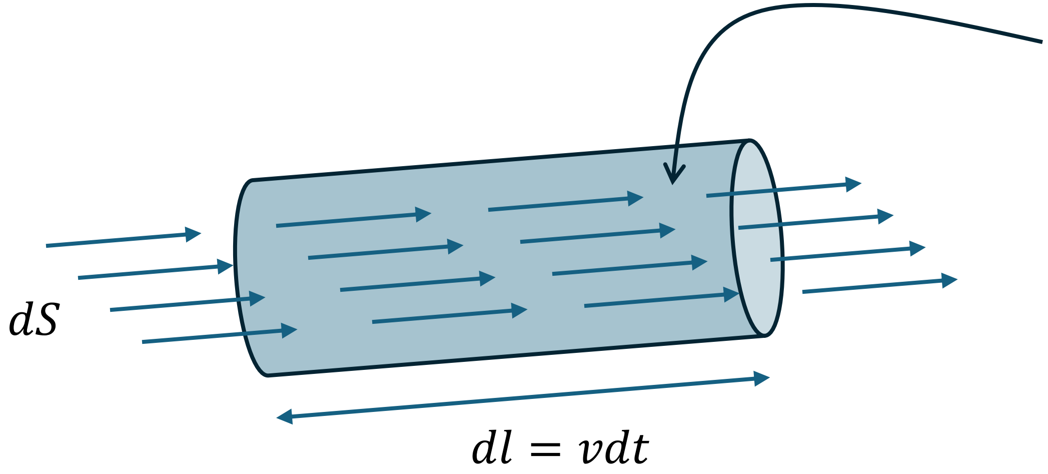
Maxwell-Ampère



"Magnetic field circulates around a moving charges or a time-varying electric field"

The surface current density

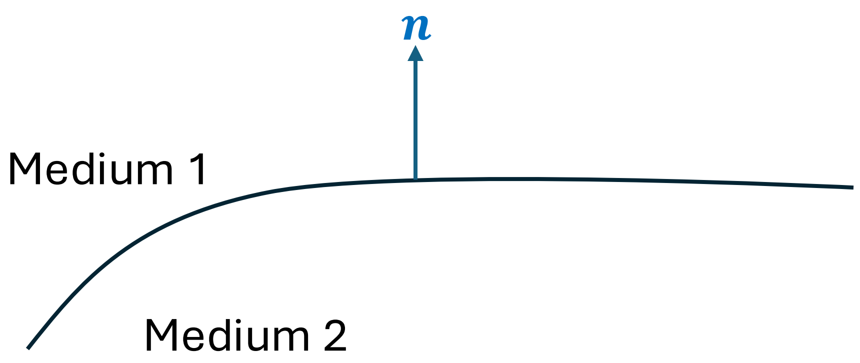
$\boldsymbol{J} = \sum_i \rho_i \boldsymbol{v_i} \delta(r - r_i)$: electric current density [A/m^2]



$$I_{tot,dS} = \frac{number\ of\ charges\ through\ dS\ in\ dt}{dt} = \frac{\rho \times dS \times dl}{dt} = \frac{\rho \times dS \times vdt}{dt} = JdS$$

Generalization: $\iint_S \boldsymbol{J}.\boldsymbol{dS} = I_{tot,S}$

Boundary conditions



$$\boldsymbol{n}.\left(\boldsymbol{D}_1 - \boldsymbol{D}_2\right) = \rho_S$$

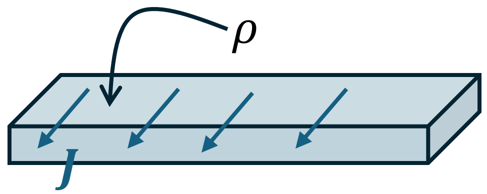
$$\boldsymbol{n}.\left(\boldsymbol{B}_1 - \boldsymbol{B}_2\right) = 0$$

$$\boldsymbol{n} \times \left(\boldsymbol{E}_1 - \boldsymbol{E}_2\right) = \boldsymbol{0}$$

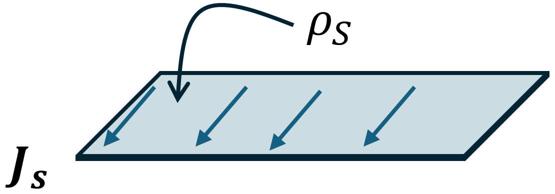
$$\boldsymbol{n} \times \left(\boldsymbol{H}_1 - \boldsymbol{H}_2\right) = \boldsymbol{J}_s$$

$$\rho_S = \lim \rho\, \Delta z\ [\Delta z \to 0]$$

$$\boldsymbol{J}_s = \lim \boldsymbol{J}\, \Delta z\ [\Delta z \to 0]$$



$[\Delta z \to 0]$
$\Longrightarrow$



Complex Maxwell equations

Only for time-harmonic solutions oscillating at the pulsation ω, $\mathbf{E(t)} = \mathrm{Re}\left[\boldsymbol{E}(\omega)e^{j\omega t}\right]$

The complex vector $\boldsymbol{E}(\omega)$ is called a phasor (it's a Fourier transform)

ex: $\mathbf{E(t)} = 2\cos(\omega t + 3z)\boldsymbol{x}$
$\boldsymbol{E}(\omega) = 2e^{j3z}\boldsymbol{x}$

When transposing the equations from real fields to phasors, temporal derivatives become products by $j\omega$:

$$\nabla.\boldsymbol{D}(\omega) = \rho(\omega)$$

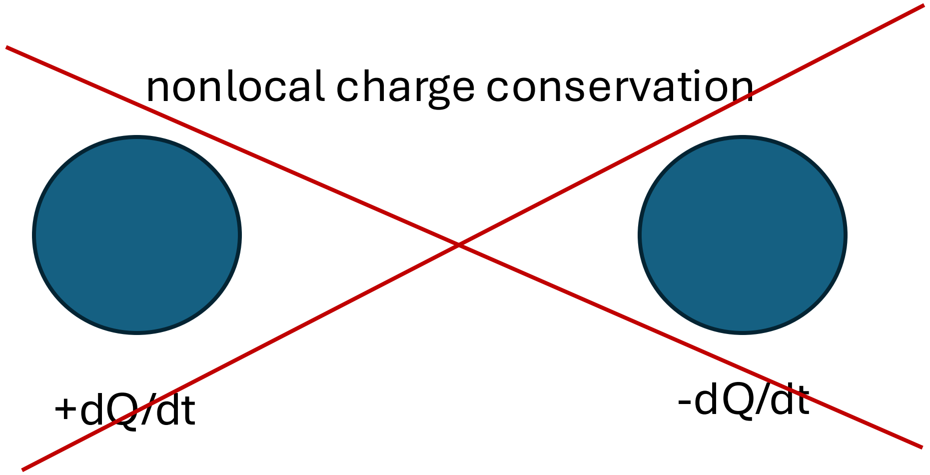
$$\nabla.\boldsymbol{B}(\omega) = 0$$

$$\nabla \times \boldsymbol{E}(\omega) = -j\omega\boldsymbol{B}(\boldsymbol{\omega})$$

$$\nabla \times \boldsymbol{H}(\omega) = \boldsymbol{J}(\omega) + j\omega\boldsymbol{D}(\omega)$$

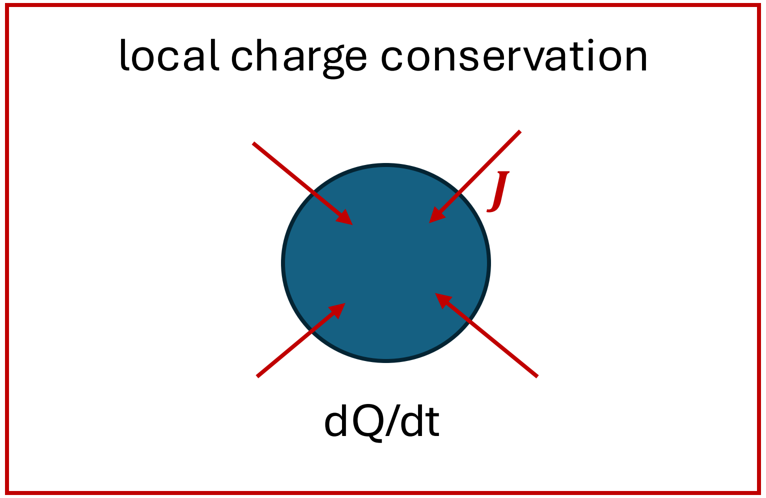
we will omit to explicit the (ω) dependency of phasors unless needed for clarity

Local charge conservation



$$\nabla . \boldsymbol{D} = \rho$$

$$\nabla . (\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t})$$



$$\boxed{\nabla . \boldsymbol{J} = -\frac{\partial \rho}{\partial t}}$$

Integral version via divergence theorem:

$$\nabla . \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \Longrightarrow \iiint_V \nabla . \boldsymbol{J}\, dV = -\frac{\partial}{\partial t} \iiint_V \rho\, dV$$

$$\Longrightarrow \iint_S \boldsymbol{J} . \boldsymbol{dS} = -\frac{\partial}{\partial t} Q_{tot}$$

The sources of the field

$\rho = \sum_i \rho_i \delta(r - r_i)$: electric charge density [C/m^3]

$\boldsymbol{J} = \sum_i \rho_i \boldsymbol{v_i} \delta(r - r_i)$: electric current density [A/m^2]

$\nabla.\boldsymbol{J} = -\frac{\partial \rho}{\partial t}$ $\quad \Longrightarrow \nabla.\boldsymbol{J} = -j\omega\rho$ $\quad \Longrightarrow \rho = \frac{j}{\omega}\nabla.\boldsymbol{J}$

ρ and $\boldsymbol{J}$ are linked: we can keep only $\boldsymbol{J}$ and « forget » about $\nabla.\boldsymbol{D} = \rho$

$$\nabla \times \boldsymbol{E}(\omega) = -j\omega\boldsymbol{B}(\boldsymbol{\omega})$$

$$\nabla \times \boldsymbol{H}(\omega) = \boldsymbol{J}(\omega) + j\omega\boldsymbol{D}(\omega)$$

The two curl equations contain the important information about the dynamics. The divergence equations contain information about charge densities only.

Magnetic charges and currents

Generalized Maxwell's equations:

$$\nabla.\boldsymbol{D} = \rho \ = \rho_e$$

$$\nabla.\boldsymbol{B} = 0 \ = \rho_m$$

$$\nabla \times \boldsymbol{E} = -j\omega\boldsymbol{B} = -\boldsymbol{M} - j\omega\boldsymbol{B}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\boldsymbol{D}$$

$$\nabla.\boldsymbol{J} = -j\omega\rho_e$$

$$\nabla.\boldsymbol{M} = -j\omega\rho_m$$

Duality= form of symmetry between electricity/magnetism

= invariance of Maxwell's equations on the transformation:

Electric quantity ($\boldsymbol{D}$, $\boldsymbol{E}$, ρ_e, $\boldsymbol{J}$)→ Magnetic quantity ($\boldsymbol{B}$,$\boldsymbol{H}$, ρ_m, $\boldsymbol{M}$)

Magnetic quantity ($\boldsymbol{B}$,$\boldsymbol{H}$, ρ_m, $\boldsymbol{M}$) → − Electric quantity ($-\boldsymbol{D}$, $-\boldsymbol{E}$, -ρ_e, -$\boldsymbol{J}$)

« Once a solution of Maxwell's equation is found, another one is found but in the dual world»
NB: The dual world has different (dual) sources, fields and materials. It represents a different electromagnetic situation.

From vacuum to materials

Vacuum

$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}, \varepsilon_0 = 8.85\ 10^{-12}\ [F/m]$

$\boldsymbol{B} = \mu_0 \boldsymbol{H}, \mu_0 = 4\pi\ 10^{-7}\ [H/m]$

$\rho = \rho_{ext} = \rho_{source}$

$\boldsymbol{J} = \boldsymbol{J}_{ext} = \boldsymbol{J}_{source}$

Non-dispersive dielectrics

$\boldsymbol{D} = \varepsilon \boldsymbol{E}$

$\boldsymbol{B} = \mu \boldsymbol{H}$

$\rho = \rho_{ext} = \rho_{source}$

$\boldsymbol{J} = \boldsymbol{J}_{ext} = \boldsymbol{J}_{source}$

Metals

$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$

$\boldsymbol{B} = \mu_0 \boldsymbol{H}$

$\boldsymbol{J}_{ind} = \sigma \boldsymbol{E}$

$\rho = \rho_{source}$

$\boldsymbol{J} = \boldsymbol{J}_{source} + \boldsymbol{J}_{ind}$

Dispersive dielectrics

$\boldsymbol{D}(\omega) = \varepsilon(\omega) \boldsymbol{E}(\omega)$

$\boldsymbol{B}(\omega) = \mu(\omega) \boldsymbol{H}(\omega)$

$\rho(\omega) = \rho_{ext} = \rho_{source}$

$\boldsymbol{J}(\omega) = \boldsymbol{J}_{ext} = \boldsymbol{J}_{source}$

Alternative description of metals

Metals

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

$$\boldsymbol{J}_{ind} = \sigma \boldsymbol{E}$$

$$\rho = \rho_{source}$$

$$\boldsymbol{J} = \boldsymbol{J}_{source} + \boldsymbol{J}_{ind}$$

description 1:
conductivity model

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \boldsymbol{J} + j\omega\varepsilon_0 \boldsymbol{E} \\ &= \boldsymbol{J}_{source} + \boldsymbol{J}_{ind} + j\omega\varepsilon_0 \boldsymbol{E} \\ &= \boldsymbol{J}_{source} + \sigma \boldsymbol{E} + j\omega\varepsilon_0 \boldsymbol{E} \\ &= \boldsymbol{J}_{source} + j\omega\left(\varepsilon_0 - j\frac{\sigma}{\omega\varepsilon_0}\right)\boldsymbol{E} \\ &= \boldsymbol{J}_{source} + j\omega\varepsilon_{eff}(\omega)\boldsymbol{E} \end{aligned}$$

Metals

$$\boldsymbol{D} = \varepsilon_{eff}(\omega)\boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

$$\varepsilon_{eff}(\omega) = \varepsilon_0 - j\frac{\sigma}{\omega\varepsilon_0}$$

$$\rho = \rho_{source}$$

$$\boldsymbol{J} = \boldsymbol{J}_{source}$$

description 2:
dispersive lossy
dielectric model

Both descriptions are equivalent and used.
In the second description as dielectric and metals are described by the same Maxwell equations and we can transpose results from dielectrics to metals by assuming complex permittivity.
In the second description more complex permittivity dispersions are allowed (Drude-Lorentz, etc)

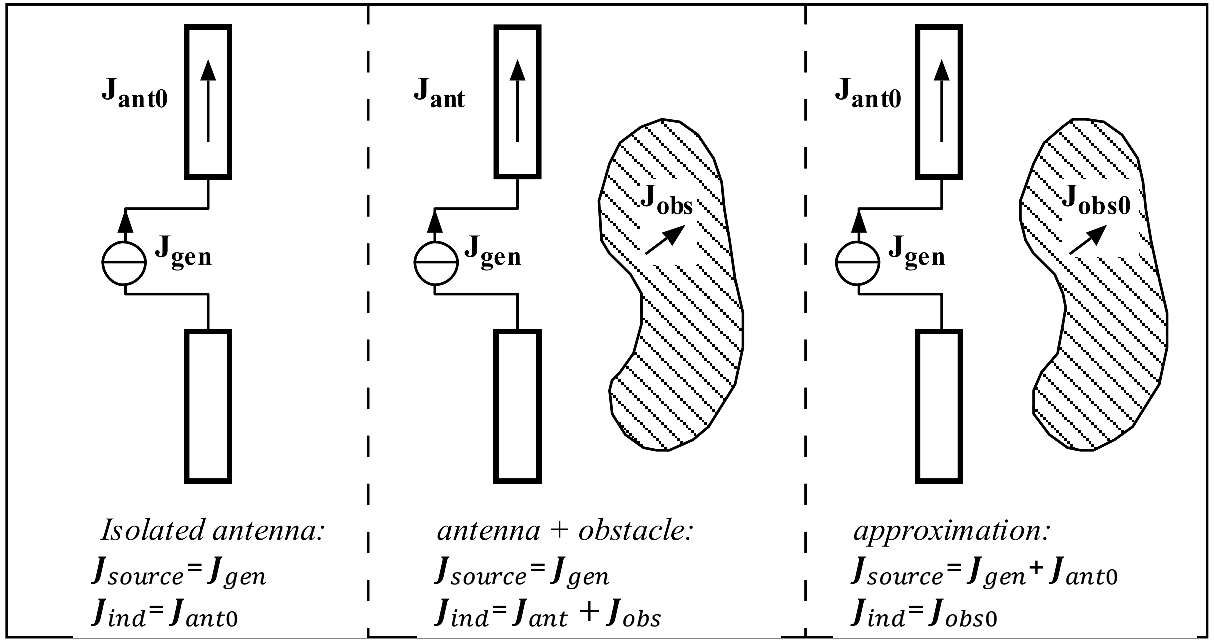
Application: Complex Maxwell equations for antennas

$$\nabla . \boldsymbol{D} = \rho_{source}$$

$$\nabla . \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{source} + j\omega\varepsilon(r,\omega)\boldsymbol{E}$$



- $\varepsilon(r,\omega)$ is a distribution of permittivity describing the antenna (metals with induced currents, dielectric, etc) and its environment (also possibly with induced currents).

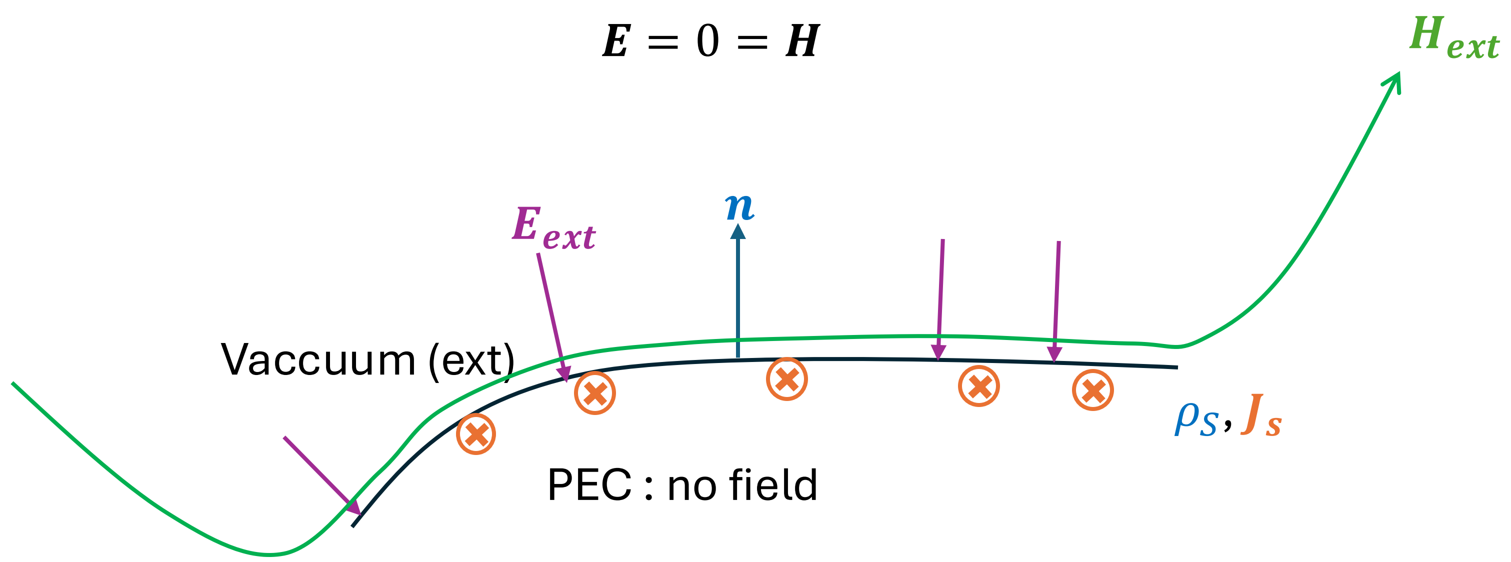
- The source currents (and charge) distribution are usually non-zero only on a small portion of the antenna called the port (like the end of the coaxial cable or small wires were the generator is applied).

- But other choices are possible especially when making approximations of the currents around the port and on the antenna.

Perfect electrical conductors (PEC)

$$\boldsymbol{J}_{ind} = \sigma \boldsymbol{E} \text{ with } \sigma \to \infty$$

$$\boldsymbol{E} = 0 = \boldsymbol{H}$$



$$\boldsymbol{n} \times (\boldsymbol{E}_{ext} - \boldsymbol{E}_{in}) = \boldsymbol{n} \times \boldsymbol{E}_{ext} = \boldsymbol{0}$$

$$\boldsymbol{n}.(\boldsymbol{D}_{ext} - \boldsymbol{D}_{in}) = \boldsymbol{n}.\boldsymbol{D}_{ext} = \rho_S$$

$$\boldsymbol{n}.(\boldsymbol{B}_{ext} - \boldsymbol{B}_{in}) = \boldsymbol{n}.\boldsymbol{B}_{ext} = 0$$

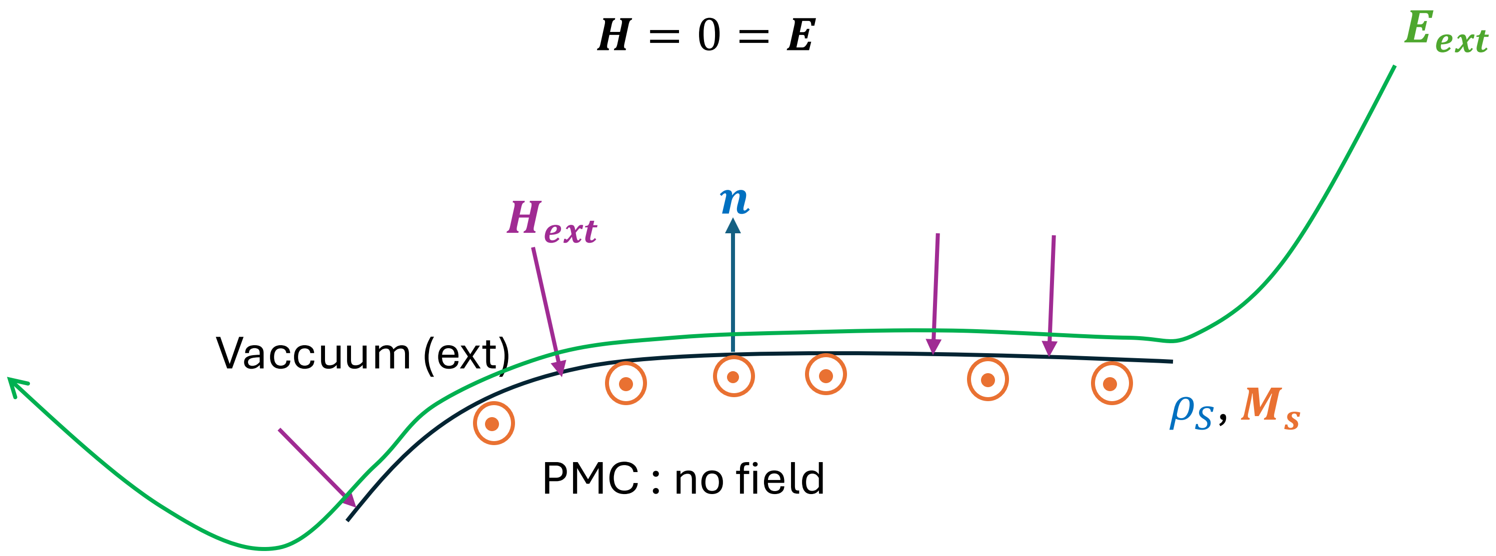
$$\boldsymbol{n} \times (\boldsymbol{H}_{ext} - \boldsymbol{H}_{in}) = \boldsymbol{n} \times \boldsymbol{H}_{ext} = \boldsymbol{J}_s$$

Perfect magnetic conductors (PMC)

The dual of Perfect Electric conductors

$\boldsymbol{M}_{ind} = \sigma \boldsymbol{H}$ with $\sigma \rightarrow \infty$

$\boldsymbol{H} = 0 = \boldsymbol{E}$



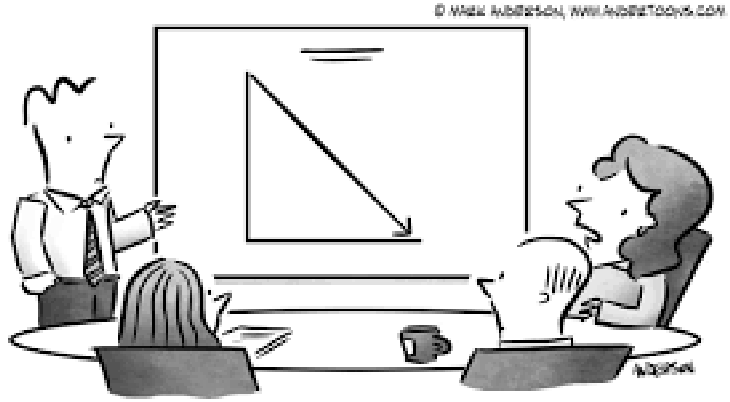
$\boldsymbol{n} \times (\boldsymbol{H}_{ext} - \boldsymbol{H}_{in}) = \boldsymbol{n} \times \boldsymbol{H}_{ext} = \boldsymbol{0}$

$\boldsymbol{n}.(\boldsymbol{B}_{ext} - \boldsymbol{B}_{in}) = \boldsymbol{n}.\boldsymbol{B}_{ext} = \rho_{mS}$

$\boldsymbol{n}.(\boldsymbol{D}_{ext} - \boldsymbol{D}_{in}) = \boldsymbol{n}.\boldsymbol{D}_{ext} = 0$

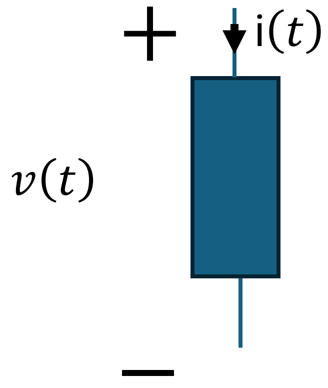
$\boldsymbol{n} \times (\boldsymbol{E}_{ext} - \boldsymbol{E}_{in}) = \boldsymbol{n} \times \boldsymbol{H}_{ext} = -\boldsymbol{M}_s$

2) A couple useful theorems



"Yes, we can apply the Pythagorean Theorem, but I'm more interested in *why* this is happening."

Instantaneous and average power in a circuit



Harmonic signals:

$v(t) = v_0 \cos(\omega t + \varphi_v)$

$\mathrm{i}(t) = i_0 \cos(\omega t + \varphi_i)$

$\mathrm{P}(t) = v(t)\,\mathrm{i}(t)$

Complex Phasors:

$V = v_0 e^{j\varphi_v}$

$I = i_0 e^{j\varphi_i}$

$$P_{av} = \int_0^{T=2\pi/\omega} P(t) = \frac{1}{2} v_0 i_0 \cos(\varphi_v - \varphi_i) = \frac{1}{2} Re(VI^*)$$

Power absorbed/provided to the charges

Instantaneous and average power (density) in the field

Harmonic signals:

$\boldsymbol{E}(\boldsymbol{t}) = \boldsymbol{E_0}\cos(\omega t + \varphi_E)$

$\boldsymbol{J}(\boldsymbol{t}) = \boldsymbol{J_0}\cos(\omega t + \varphi_J)$

$\mathrm{p}(t) = \boldsymbol{E}(\boldsymbol{t}).\boldsymbol{J}(\boldsymbol{t})$

Complex Phasors:

$\boldsymbol{E}= \boldsymbol{E_0}e^{j\varphi_E}$

$\boldsymbol{J}= \boldsymbol{J_0}e^{j\varphi_J}$

$$-p_{av}= -\int_0^{T=\frac{2\pi}{\omega}} P(t) = -\frac{1}{2}\boldsymbol{E_0}.\boldsymbol{J_0}\cos(\varphi_v - \varphi_i) = -\frac{1}{2}Re(\boldsymbol{E}.\boldsymbol{J}^*)$$

Power provided by the charges to the field= -Power absorbed/provided to the charges by the field

Complex Poynting theorem

$$p_{av} = -\frac{1}{2} Re(\boldsymbol{E}.\boldsymbol{J}^*)$$

$$(\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B}\ ).\boldsymbol{H}^*$$

$$-\ (\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega \boldsymbol{D})^*.\boldsymbol{E}$$

$$\boldsymbol{H}^*.\nabla \times \boldsymbol{E} - \boldsymbol{E}.\nabla \times \boldsymbol{H}^* = \nabla.(\boldsymbol{E} \times \boldsymbol{H}^*)$$

$$\nabla.(\boldsymbol{E} \times \boldsymbol{H}^*) = -j\omega \boldsymbol{B}.\boldsymbol{H}^* + j\omega \boldsymbol{E}.\boldsymbol{D}^* - \boldsymbol{E}.\boldsymbol{J}^*$$

Assume free-space:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

$$\Rightarrow \frac{1}{2}\nabla.(\boldsymbol{E} \times \boldsymbol{H}^*) = -\frac{j\omega}{\mathbf{2}}\left(\mu_0 \boldsymbol{H}.\boldsymbol{H}^* - \varepsilon_0 \boldsymbol{E}.\boldsymbol{E}^*\right) - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{E}.\boldsymbol{J}^*$$

Complex Poynting theorem

Real part : real power balance
Im part: reactive power balance

Take the real part: $p_{av} = -\frac{1}{2} Re(\boldsymbol{E}.\boldsymbol{J}^*) = \frac{1}{2}\nabla.\mathrm{Re}(\boldsymbol{E} \times \boldsymbol{H}^*)$

$$\boldsymbol{S} = Re\left[\frac{1}{2}(\boldsymbol{E} \times \boldsymbol{H}^*)\right]$$

Poynting vector (W/m²)

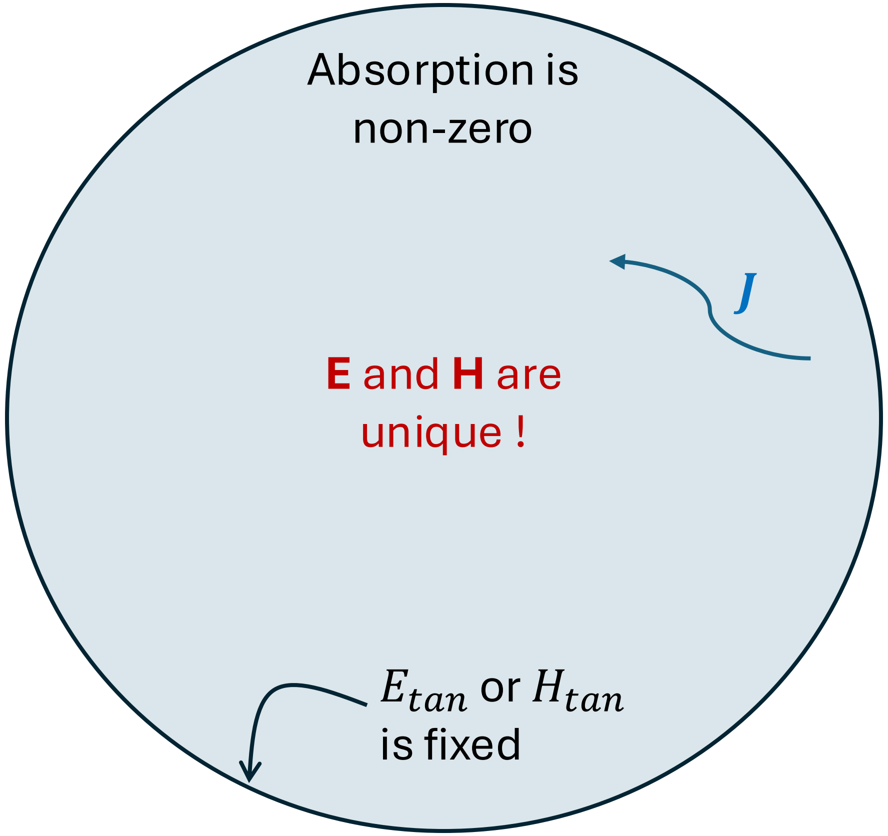
Integrate over volume: $P_{av} = -\int_{\boldsymbol{V}} \frac{1}{2} Re(\boldsymbol{E}.\boldsymbol{J}^*) = \oint_{\boldsymbol{S}} Re\left[\frac{1}{2}(\boldsymbol{E} \times \boldsymbol{H}^*)\right].\boldsymbol{dS}$

Power provided by the currents in V

Power oozing out of S

Unicity

From Maxwell's equations, we can prove that (see e.g. EE-201 for a demo):

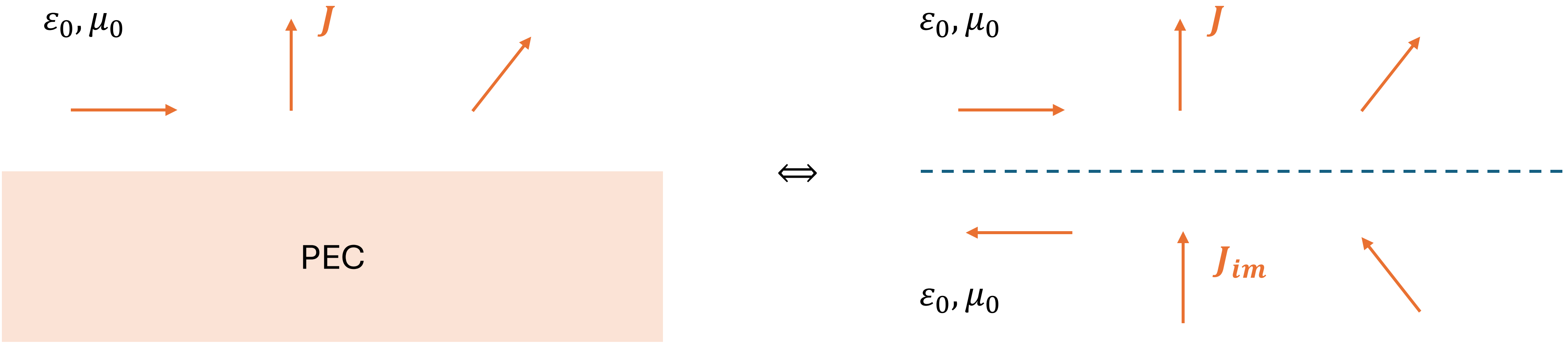


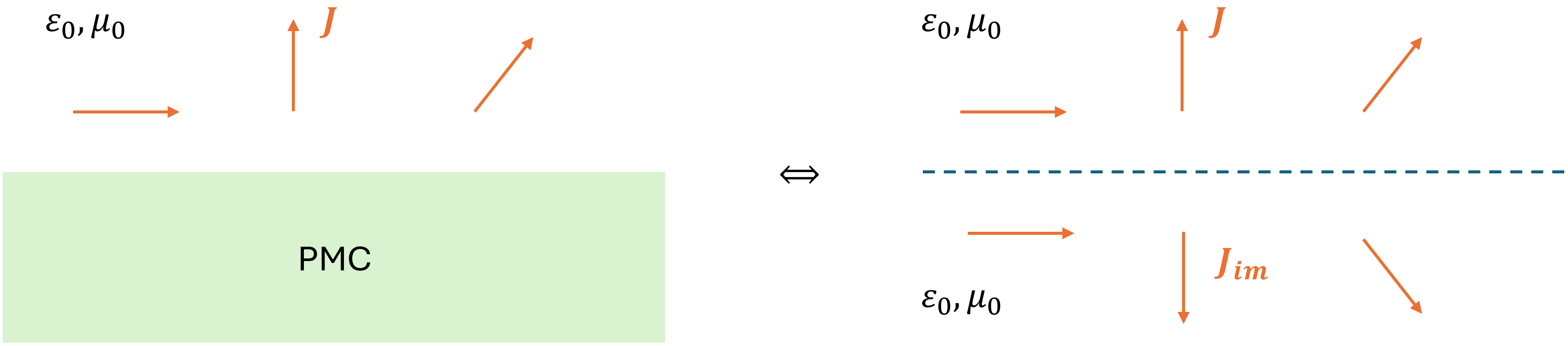
The solutions to Maxwell equations is unique in a volume V iif:

1) Proper boundary conditions are set. Any of the following:
   - The tangential E field is specified on the boundary S of the volume
   - The tangential H field is specified on the boundary S of the volume
   - The tangential E field is specified in part of the boundary S of the volume, and the tangential H is specified on the rest of the boundary.

2) There exist absorption in the volume V, at least a tiny bit.

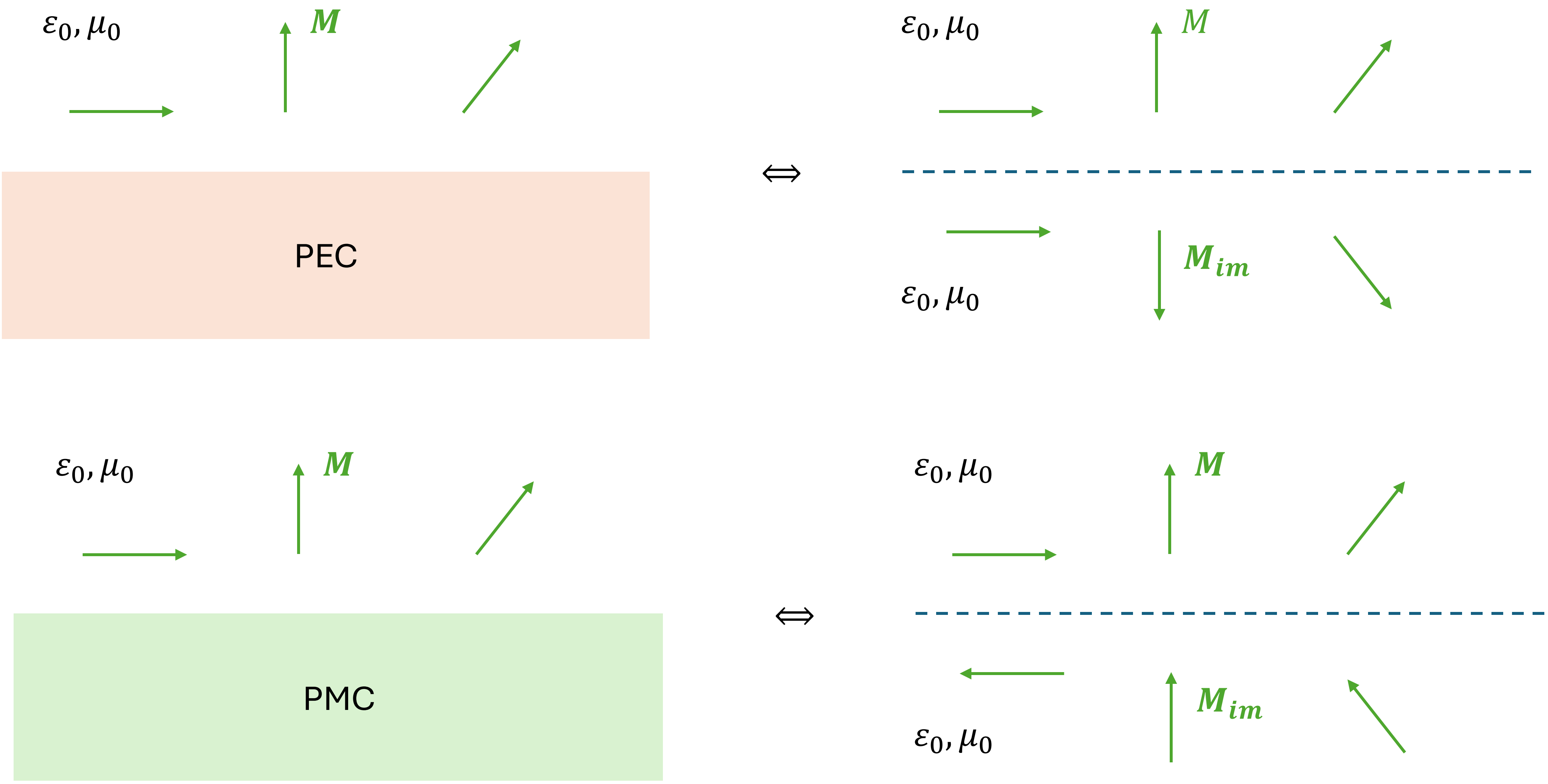
Image theorems for electric currents





Proof: use the unicity theorem !

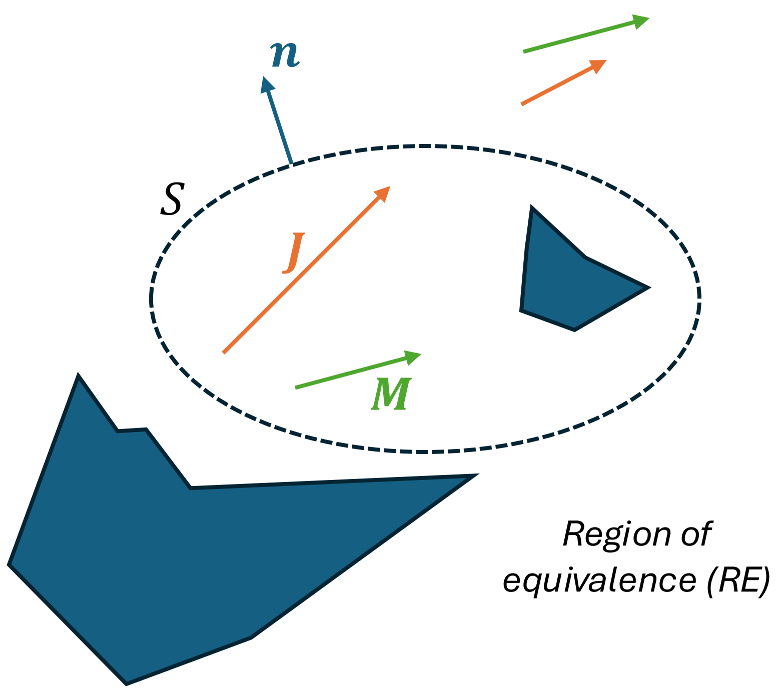
Image theorems for magnetic currents



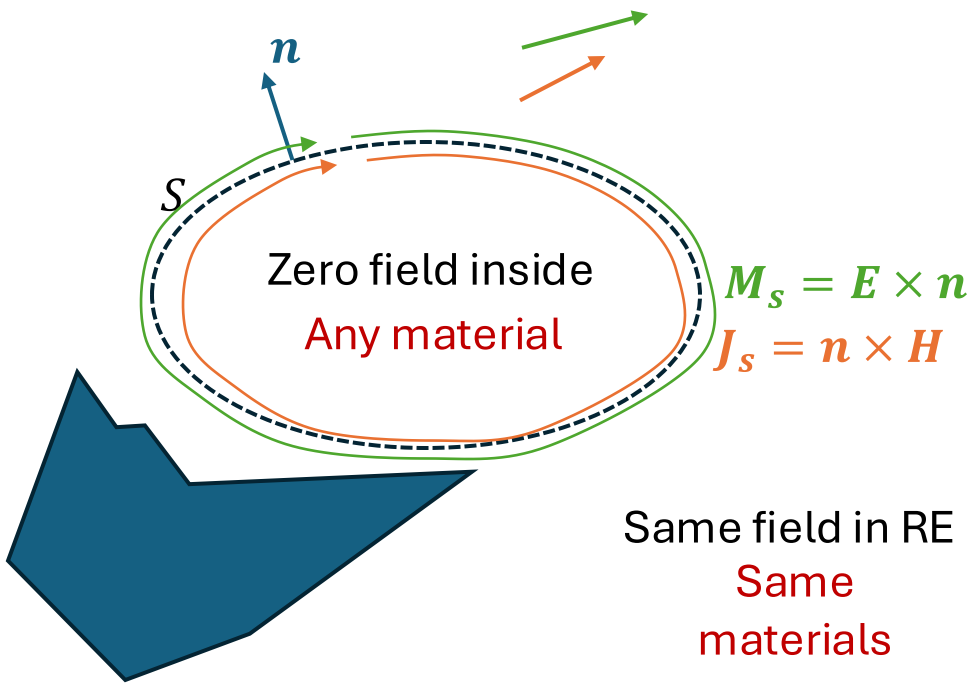
Proof: use the unicity theorem !

Huyghens equivalence principle

Initial problem



Equivalent problem



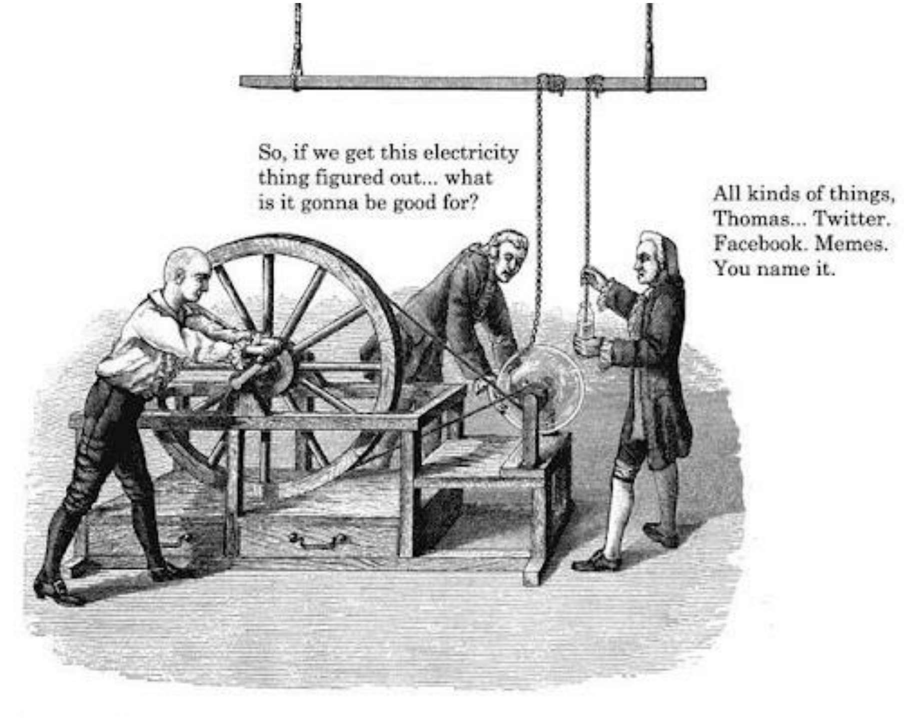
1) Choose S, put n towards RE

2) Cut and paste everything in the RE

3) Put the Hughens sources on S

4) Choose whatever material(s) inside S (vaccuum PEC, PMC, your choice)

3) Solving the equations…
…in a homogenous medium



Source-free solutions: the case of plane waves

Free space $\varepsilon, \mu \in \mathbb{R}^*$

$$\nabla \times \left(\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \right)$$
$$\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E}$$

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla . \boldsymbol{E}) - \nabla^2 \boldsymbol{E} \quad (\nabla . \boldsymbol{E} = 0)$$

$$\Longrightarrow \nabla^2 \boldsymbol{E} + \omega^2 \mu\varepsilon \boldsymbol{E} = 0$$ Helmholtz equation

Plane-wave solutions: $\boldsymbol{E} = \boldsymbol{e}\, E_0\, e^{-j\boldsymbol{k}.\boldsymbol{r}}$ $\quad \boldsymbol{H} = \frac{1}{Z_0}(\boldsymbol{k} \times \boldsymbol{E})$ with $\boldsymbol{e}.\boldsymbol{k} = 0, \boldsymbol{k}.\boldsymbol{k} = \omega^2 \mu\varepsilon$ and $Z_0 = \sqrt{\frac{\mu}{\varepsilon}}$

Example: $\boldsymbol{E} = (\boldsymbol{x}E_{0x} + \boldsymbol{y}E_{0y})e^{-jkz}$, with $E_{0x}, E_{0y} \in \mathbb{C}$

- If E_{0x}, E_{0y} are in phase or anti-phase: linear polarization
- If $|E_{0x}| = |E_{0y}|$, and are in phase quadrature ($\pm\pi/2$): circular polarization (RH or LH)
- General case: elliptical polarization
  - Right handed part: $E_R = \frac{1}{2}\left(E_{0x} + jE_{0y}\right) = |E_R|e^{j\theta_R}$
  - Left handed part: $E_L = \frac{1}{2}(E_{0x} - jE_{0y}) = |E_L|e^{j\theta_L}$
  - aspect ratio: $\frac{a}{b} = \frac{|E_R|+|E_L|}{||E_R|-|E_L||}$
  - inclination angle wrt x-axis: $\varphi = \frac{\theta_R - \theta_L}{2}$

$$\boldsymbol{E} = \boldsymbol{e}\, E_0\, e^{-j\boldsymbol{k}.\boldsymbol{r}}$$

The dispersion relation

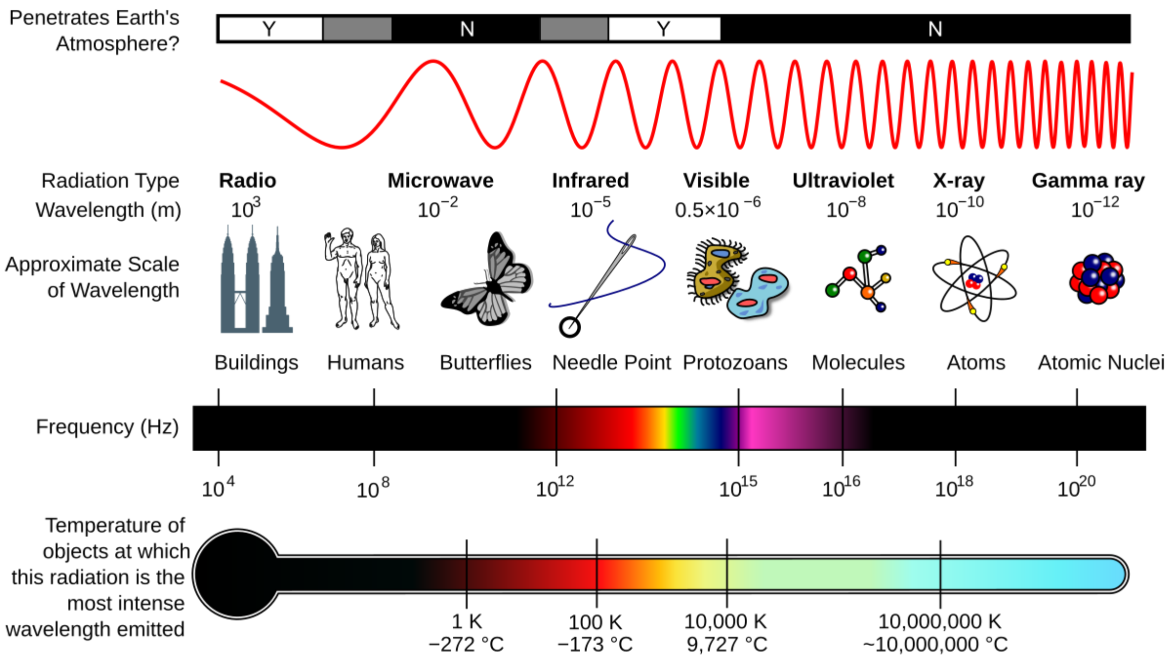
$$\boxed{\boldsymbol{k}.\boldsymbol{k} = \omega^2\, \mu\varepsilon}$$

$$\boldsymbol{k} = k\boldsymbol{z} \Longrightarrow k = \pm\omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}$$

$$v_\varphi = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$k = \omega\sqrt{\mu\varepsilon} = 2\pi f\, \frac{1}{v_\varphi} = \frac{2\pi}{\lambda}$$

$$\boxed{\Longrightarrow \lambda = \frac{v_\varphi}{f}}$$



GPS: 1.5 GHz

Wi-fi: 2.4GHz /5 GHz

The Vector Potential

$= not\ 0$ anymore ! = complicated

Free space
$\varepsilon, \mu \in \mathbb{R}^*$

$\nabla \times ($ $\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H})$
$\nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} + \boldsymbol{J}$

$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla.\boldsymbol{E}) - \boldsymbol{\nabla}^2\ \boldsymbol{E}$

Alternative: Simplify things first using the *Helmoltz partition theorem (HPT)*

$\forall \boldsymbol{U}$ *sufficiently well-behaved (smooth and decaying, simply connected space):*
$\exists\ (\boldsymbol{V}, W)$ *such that* $\boldsymbol{U} = \boldsymbol{\nabla} \times \boldsymbol{V} - \boldsymbol{\nabla} W$

$$W(\boldsymbol{r}) = \frac{1}{4\pi}\int d^3\boldsymbol{r}' \frac{\nabla'.\boldsymbol{U}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad V(\boldsymbol{r}) = \frac{1}{4\pi}\int d^3\boldsymbol{r}' \frac{\nabla' \times \boldsymbol{U}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

And a corrolary (HPTc):
To define such a well-behaved field $\boldsymbol{U}$ *it is required to fix both* $\boldsymbol{\nabla} \times \boldsymbol{U}$ *and* $\boldsymbol{\nabla}.\boldsymbol{U}$

Apply HPT to $\boldsymbol{B}$*:* $\exists\ (\boldsymbol{A}, W)$ *such that* $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} - \boldsymbol{\nabla} W.$ But since $\nabla.B = 0$, then $W = 0$.

Conclusion: $\exists\ \boldsymbol{A}$ such that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. $\boldsymbol{A}$ is not yet uniquely defined, we still need to fix $\boldsymbol{\nabla}.\boldsymbol{A}$.
Fixing $\nabla.\boldsymbol{A}$ is called making a Gauge choice. $\boldsymbol{A}$ is called the vector potential

The scalar potential

Free space
$\varepsilon, \mu \in \mathbb{R}^*$

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} = -j\omega(\boldsymbol{\nabla} \times \boldsymbol{A})$$
$$\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J}$$

$$\nabla \times (\boldsymbol{E} + j\omega \boldsymbol{A}) = \boldsymbol{0}$$

$\boldsymbol{E} + j\omega \boldsymbol{A}$ is a pure gradient ! (by TPH or, if one prefers, by Poincaré lemma)

Conclusion: $\exists\ \Phi$, defined up to a constant, such that $\boldsymbol{E} + j\omega \boldsymbol{A} = -\nabla\Phi$.
Φ is called the scalar potential. It becomes the electrostatic potential when $\omega = 0$.

$\boldsymbol{E}, \boldsymbol{H}$= 6 variables → $\Phi, \boldsymbol{A}$= 4 variables

Lorentz gauge

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$
$\boldsymbol{\nabla}.\boldsymbol{A}$=TBD
$\boldsymbol{E} = -\nabla\Phi - j\omega \boldsymbol{A}$

Free homogenous space
$\varepsilon, \mu \in \mathbb{R}^*$

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B} = -j\omega(\boldsymbol{\nabla} \times \boldsymbol{A})$$

$$\nabla \times \boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} + \boldsymbol{J}$$

$$\Longrightarrow \nabla \times \nabla \times \boldsymbol{A} = \omega^2 \mu\varepsilon \boldsymbol{A} - j\omega\mu\varepsilon\nabla\Phi + \mu \boldsymbol{J}$$

$$\Longrightarrow \nabla(\nabla.\boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} = k^2 \boldsymbol{A} - j\omega\mu\varepsilon\nabla\Phi + \mu \boldsymbol{J}$$

Choose $\boldsymbol{\nabla}.\boldsymbol{A} = -j\omega\mu\varepsilon\Phi$ (Lorentz Gauge) !

$$\Longrightarrow \boldsymbol{\nabla}^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

Helmholtz equation

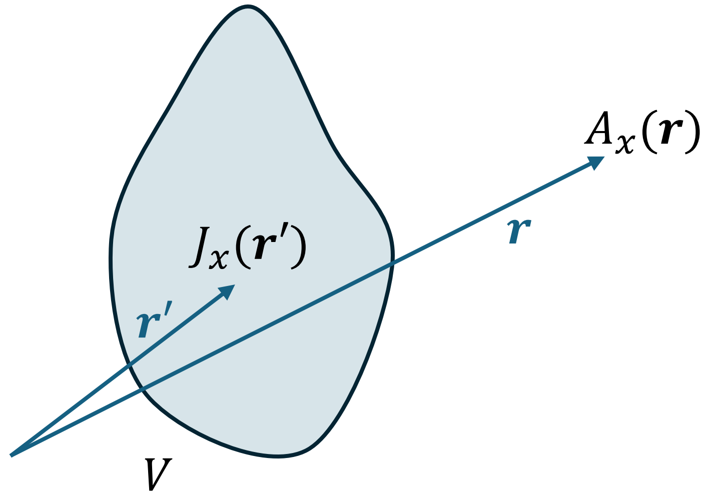
We get three uncoupled Helmholtz equations !

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$
$$\nabla^2 A_y + k^2 A_y = -\mu J_y$$
$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

How to solve ? do J_x alone first

$\nabla^2 A_x + k^2 A_x = -\mu J_x$

Green's function is all you need



Let $g(\boldsymbol{r}, \boldsymbol{r}')$ be the vector potential created at $\boldsymbol{r}$ by a unity current source located at $\boldsymbol{r}'$:

$$\nabla^2 g(\boldsymbol{r}, \boldsymbol{r}') + k^2 g(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}')$$

By linearity, we would then have:

$$A_x(\boldsymbol{r}) = \iiint_V d^3\boldsymbol{r}' \mu J_x(\boldsymbol{r}') g(\boldsymbol{r}, \boldsymbol{r}')$$

All we need is $g(\boldsymbol{r}, \boldsymbol{r}')$. We will show in the exercise session that:

$$g(\boldsymbol{r}, \boldsymbol{r}') = g(|\boldsymbol{r} - \boldsymbol{r}'|) = \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

Summary: exact solutions of Maxwells equations

1) Start from a known current distribution $\boldsymbol{J}(\boldsymbol{r}')$

2) Compute the vector potential: $\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_V J_x(\boldsymbol{r}')\frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3\boldsymbol{r}'$

3) Calculate $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$

4) Calculate $\Phi = \frac{j}{\omega\mu\varepsilon}\nabla.\boldsymbol{A}$

5) Calculate $\boldsymbol{E} = -\nabla\Phi - j\omega\boldsymbol{A}$

Exercise: The exact field radiated by an infinitesimal dipole